

GLOBAL STRONG SOLUTIONS TO INCOMPRESSIBLE ERICKSEN-LESLIE SYSTEM IN \mathbb{R}^3

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ABSTRACT. In this paper, we consider the Cauchy problem to the Ericksen-Leslie system of liquid crystals in \mathbb{R}^3 . Global well-posedness of strong solutions are obtained under the condition that the product of $\|u_0\|_2 + \|\nabla d_0\|_2$ and $\|\nabla u_0\|_2 + \|\nabla^2 d_0\|_2$ is suitably small. This result can be viewed as a supplement to the local existence and blow up criteria discussed in [9].

1. INTRODUCTION

Continuum theory for nematic liquid crystals was initiated by Oseen[15] in static version, and was reformulated by Frank[6]. Ericksen[4, 5] and Leslie[11, 12] proposed the corresponding dynamic model by extending their work, and the model can be successfully used to model the situation without defects. The Oseen-Frank free energy of liquid crystal occupied in region $\Omega \subset \mathbb{R}^3$ with a configuration $d \in H^1(\Omega; S^2)$ is

$$E(d; \Omega) = \int_{\Omega} W(d, \nabla d) dx,$$

where the Oseen-Frank density is given by

$$W(d, \nabla d) = k_1(\operatorname{div} d)^2 + k_2(d \cdot \operatorname{curl} d)^2 + k_3|d \times \operatorname{curl} d|^2$$

for positive constants $k_i, i = 1, 2, 3$. One can refer to [7, 13] for more details on static theory of liquid crystals.

In general, the incompressible Ericksen-Leslie model reads as follows,

$$\partial_t u^i + (u \cdot \nabla) u^i - \Delta u^i + \partial_i P = -\partial_j (\partial_i d^k W_{p_j^k}(d, \nabla d)), \quad (1.1a)$$

$$\nabla \cdot u = 0, \quad |d| = 1, \quad (1.1b)$$

$$\begin{aligned} \partial_t d^i + (u \cdot \nabla) d^i &= \partial_j (W_{p_j^i}(d, \nabla d)) - W_{d^i}(d, \nabla d) \\ &\quad - (\partial_j (W_{p_j^k}(d, \nabla d)) - W_{d^k}(d, \nabla d)) d^k d^i, \end{aligned} \quad (1.1c)$$

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where $u = (u^1, u^2, u^3)$ is the velocity of the fluid, $d = (d^1, d^2, d^3)$ is the unit molecular direction, and P is the pressure. Additionally, $i, j, k = 1, 2, 3$, and the Einstein summation is used.

The above Ericksen-Leslie model is a coupled system by Navier-Stokes equations and the gradient flow of the Oseen-Frank model. Both of their developments are heuristic to our further discussion. The well-known result on Navier-Stokes equations is about the existence and partial regularity of global suitable weak solutions presented in [2, 14], and the Serrin or Beale-Kato-Majda type blow up criteria presented in [1, 16]. The development related to the heat flow of harmonic maps, which is a specific example of the above gradient flow, is the existence and partial regularity of global weak solutions presented in [3, 17]. Accordingly, many literatures studied the simplified version of nematic liquid crystal equations, where required $k_1 = k_2 = k_3$. In particular, the local strong solutions were obtained in [18], and the blow up criteria in [10]. For the more general case, [8] started the study on existence and regularity in \mathbb{R}^2 , and [9] on local existence and blow up criteria in \mathbb{R}^3 .

As discussed in [9], we consider the Cauchy problem to the general Ericksen-Leslie system above in \mathbb{R}^3 . The following initial data are imposed to (1.1):

$$(u, d)|_{t=0} = (u_0, d_0). \quad (1.2)$$

Moreover, we always suppose without any further mention that the initial data u_0 and d_0 satisfy

$$u_0 \in H_\sigma^1(\mathbb{R}^3), \quad d_0 - d_0^* \in H^2(\mathbb{R}^3), \quad |d_0| = 1, \quad (1.3)$$

where d_0^* is a constant unit vector, and $H_\sigma^1(\mathbb{R}^3) = \{v \in H^1(\mathbb{R}^3) \mid \operatorname{div} v = 0\}$.

Throughout this paper, we use C for a generic positive constant which may change from line to line, $\|\cdot\|_q$ for the $L^q(\mathbb{R}^3)$ norm with $q \geq 1$, and $\int |\cdot| dx$ for $\int_{\mathbb{R}^3} |\cdot| dx$.

This paper is devoted to global existence of strong solutions under certain smallness conditions. For convenience, we present definitions related to global strong solutions before statement of our main result.

Definition 1.1. *Given $T > 0$, a couple (u, d) is a strong solution to system (1.1)-(1.2) on $\mathbb{R}^3 \times (0, T)$, if it has the regularity properties*

$$u, \nabla d \in C([0, T]; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)),$$

$$\partial_t u \in L^2(0, T; L^2(\mathbb{R}^3)), \quad \partial_t d \in L^2(0, T; H^1(\mathbb{R}^3));$$

and it satisfies (1.1) a.e. on $\mathbb{R}^3 \times (0, T)$, and the initial condition (1.2).

Definition 1.2. *A finite positive number \mathcal{T} is called the maximal existence time of a strong solution (u, d) to system (1.1)-(1.2) on $\mathbb{R}^3 \times (0, T)$, if for any $T < \mathcal{T}$, (u, d) is a strong solution to system (1.1)-(1.2) on $\mathbb{R}^3 \times (0, T)$, and*

$$\lim_{T \rightarrow \mathcal{T}^-} \sup_{0 \leq t \leq T} (\|u(t)\|_{H^1}^2 + \|\nabla d(t)\|_{H^1}^2) = \infty.$$

Definition 1.3. A couple (u, d) is called a global strong solution to system (1.1)-(1.2) on $\mathbb{R}^3 \times (0, \infty)$, if it is a strong solution to system (1.1)-(1.2) on $\mathbb{R}^3 \times (0, T)$ for any finite time T .

Theorem 1.1. Under the condition (1.3), system (1.1)-(1.2) has a unique global strong solution, provided

$$(\|u_0\|_2 + \|\nabla d_0\|_2)(\|\nabla u_0\|_2 + \|\nabla^2 d_0\|_2) \leq \varepsilon_0,$$

where ε_0 is a small positive constant depending only on k_1, k_2, k_3 .

Remark 1.1. $(\|u\|_2 + \|\nabla d\|_2)(\|\nabla u\|_2 + \|\nabla^2 d\|_2)$ is a scaling invariance under the transform

$$u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad d_\lambda(x, t) = d(\lambda x, \lambda^2 t).$$

Therefore, our result can be viewed as the global existence of strong solutions in critical space.

2. PROOF OF THEOREM 1.1

Recalling the expression of $W(d, \nabla d)$ in the introduction, one can easily check that

$$W(z, p) \geq a|p|^2, \quad W_{p_\alpha^i p_\beta^j}(z, p) \xi_\alpha^i \xi_\beta^j \geq a|\xi|^2,$$

for any $z \in \mathbb{R}^3, p, \xi \in \mathbb{M}^{3 \times 3}$, where $a = \min\{k_1, k_2, k_3\}$, and

$$\begin{aligned} |W(d, \nabla d)| &\leq C|d|^2|\nabla d|^2, & |W_{d^i}(d, \nabla d)| &\leq C|d||\nabla d|^2, \\ |W_{d^i d^j}(d, \nabla d)| &\leq C|\nabla d|^2, & |W_{p_\alpha^i}(d, \nabla d)| &\leq C|d|^2|\nabla d|, \\ |W_{p_\alpha^i p_\beta^j}(d, \nabla d)| &\leq C|d|^2, & |W_{d^i p_\beta^j}(d, \nabla d)| &\leq C|d||\nabla d|, \end{aligned}$$

for a positive constant C depending only on k_1, k_2, k_3 . These inequalities will be used frequently without any further mention in this paper.

We first cite the following local existence and blow up criteria of strong solutions, which is a special case of those in Hong-Li-Xin [9].

Lemma 2.1. (Local existence and blow up criteria of strong solutions) Suppose that the condition (1.3) holds true. Then system (1.1)-(1.2) has a unique local strong solution (u, d) on $\mathbb{R}^3 \times (0, T)$, for a positive number T depending only on the initial data and k_1, k_2, k_3 .

For strong solutions, we have the following basic energy balance law.

Lemma 2.2. (Basic energy balance law, see e.g., Hong [8]) Let (u, d) be a strong solution to system (1.1)-(1.2) on $\mathbb{R}^3 \times (0, T)$. Then

$$\frac{d}{dt} \int \left(\frac{|u|^2}{2} + W(d, \nabla d) \right) dx + \int (|\nabla u|^2 + |\partial_t d + (u \cdot \nabla) d|^2) dx = 0, \quad (2.1)$$

for any $t \in (0, T)$.

However, the estimate on the second order spatial derivatives of the director field d was not included in the basic energy balance law. This estimate is given by the following lemma.

Lemma 2.3. *(First order energy inequality) Let (u, d) be a strong solution to system (1.1)–(1.2) on $\mathbb{R}^3 \times (0, T)$. Then*

$$\frac{d}{dt} \int |\nabla d|^2 dx + 2a \int |\nabla^2 d|^2 dx \leq C \int (|u|^2 + |\nabla d|^2) |\nabla^2 d| dx, \quad (2.2)$$

for any $t \in (0, T)$.

Proof. Multiplying (1.1c) by Δd^i , and integrating on \mathbb{R}^3 , then we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx + \int \partial_\alpha W_{p_\alpha}(d, \nabla d) \cdot \Delta d dx \\ &= \int \{ (u \cdot \nabla) d + W_d(d, \nabla d) + [(\partial_\alpha W_{p_\alpha}(d, \nabla d) - W_d(d, \nabla d)) \cdot d] d \} \cdot \Delta d dx \\ &= \int [((u \cdot \nabla) d + W_d(d, \nabla d)) \cdot \Delta d - (\partial_\alpha W_{p_\alpha}(d, \nabla d) - W_d(d, \nabla d)) \cdot d |\nabla d|^2] dx \\ &\leq C \int [(|u| |\nabla d| + |\nabla d|^2) |\Delta d| + (|\nabla^2 d| + |\nabla d|^2) |\nabla d|^2] dx \\ &\leq C \int [(|u|^2 + |\nabla d|^2) |\nabla^2 d| + |\nabla d|^4] dx \\ &\leq C \int (|u|^2 + |\nabla d|^2) |\nabla^2 d| dx, \end{aligned} \quad (2.3)$$

where in the last step we have used the fact $|\nabla d|^2 = -d \cdot \Delta d$.

It follows from integrating by parts that

$$\begin{aligned} \int \partial_\alpha W_{p_\alpha}(d, \nabla d) \cdot \Delta d dx &= \int \partial_\beta W_{p_\alpha}(d, \nabla d) \cdot \partial_{\alpha\beta}^2 d dx \\ &= \int [W_{p_\alpha p_\gamma}(d, \nabla d) \partial_{\gamma\beta}^2 d + W_{p_\alpha d^j}(d, \nabla d) \partial_\beta d^j] \cdot \partial_{\alpha\beta}^2 d dx \\ &\geq a \int |\nabla^2 d|^2 dx - C \int |\nabla d|^2 |\nabla^2 d| dx. \end{aligned} \quad (2.4)$$

So combining (2.3) with (2.4), we get

$$\frac{d}{dt} \int |\nabla d|^2 dx + 2a \int |\nabla^2 d|^2 dx \leq C \int (|u|^2 + |\nabla d|^2) |\nabla^2 d| dx,$$

proving the conclusion. \square

Lemma 2.4. (Second order energy inequality, see Lemma 3.2 in Hong-Li-Xin [9])
Let (u, d) be a strong solution to system (1.1)–(1.2) on $\mathbb{R}^3 \times (0, T)$. Then

$$\begin{aligned} & \frac{d}{dt} \int (|\nabla u|^2 + |\nabla^2 d|^2) dx + \int \left(|\nabla^2 u|^2 + \frac{3}{2} a |\nabla^3 d|^2 \right) dx \\ & \leq C \int (|u|^2 + |\nabla d|^2) (|\nabla u|^2 + |\nabla^2 d|^2) dx, \end{aligned}$$

for any $t \in (0, T)$.

The proof of Theorem 1.1 relies on the following two propositions.

Proposition 2.1. Let (u, d) be a strong solution to system (1.1)–(1.2) on $\mathbb{R}^3 \times (0, T)$. For any $t \in (0, T)$, define

$$m(t) = \sup_{0 \leq s \leq t} (\|u(s)\|_2 + \|\nabla d(s)\|_2) (\|\nabla u(s)\|_2 + \|\nabla^2 d(s)\|_2).$$

Then we have

$$\begin{aligned} & \frac{d}{dt} \int (|u|^2 + |\nabla d|^2 + 2W(d, \nabla d)) dx + \int (|\nabla u|^2 + a|\nabla^2 d|^2) dx \leq 0, \\ & \frac{d}{dt} \int (|\nabla u|^2 + |\nabla^2 d|^2) dx + \frac{1}{2} \int (|\nabla^2 u|^2 + a|\nabla^3 d|^2) dx \leq 0, \end{aligned}$$

for any $t \in (0, T)$, as long as $m(t) \leq \varepsilon_1$, where ε_1 is a small constant depending only on a .

Proof. By virtue of Lemma 2.2 and 2.3, we have

$$\begin{aligned} & \frac{d}{dt} \int (|u|^2 + |\nabla d|^2 + 2W(d, \nabla d)) dx + \int (2|\nabla u|^2 + 2a|\nabla^2 d|^2) dx \\ & \leq C \int (|u|^2 + |\nabla d|^2) |\nabla^2 d| dx. \end{aligned} \tag{2.5}$$

The right-hand term is then estimated by Hölder and Sobolev embedding inequalities

$$\begin{aligned} \int (|u|^2 + |\nabla d|^2) |\nabla^2 d| dx & \leq C (\|u\|_4^2 + \|\nabla d\|_4^2) \|\nabla^2 d\|_2 \\ & \leq C (\|u\|_2 + \|\nabla d\|_2)^{\frac{1}{2}} (\|u\|_6 + \|\nabla d\|_6)^{\frac{3}{2}} \|\nabla^2 d\|_2 \\ & \leq C (\|u\|_2 + \|\nabla d\|_2)^{\frac{1}{2}} (\|\nabla u\|_2 + \|\nabla^2 d\|_2)^{\frac{3}{2}} \|\nabla^2 d\|_2 \\ & \leq C (\|u\|_2 + \|\nabla d\|_2)^{\frac{1}{2}} (\|\nabla u\|_2 + \|\nabla^2 d\|_2)^{\frac{5}{2}} \\ & \leq C m(t)^{\frac{1}{2}} (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) \\ & \leq C \varepsilon_1^{\frac{1}{2}} (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) \\ & \leq \|\nabla u\|_2^2 + a \|\nabla^2 d\|_2^2, \end{aligned}$$

as long as $m(t) \leq \varepsilon_1$, and ε_1 is small enough. Thus it follows from (2.5) that

$$\frac{d}{dt} \int (|u|^2 + |\nabla d|^2 + 2W(d, \nabla d)) dx + \int (|\nabla u|^2 + a|\nabla^2 d|^2) dx \leq 0,$$

the first conclusion holds.

By virtue of Lemma 2.4, we have

$$\begin{aligned} & \frac{d}{dt} \int (|\nabla u|^2 + |\nabla^2 d|^2) dx + \int (|\nabla^2 u|^2 + \frac{3}{2}a|\nabla^3 d|^2) dx \\ & \leq C \int (|u|^2 + |\nabla d|^2)(|\nabla u|^2 + |\nabla^2 d|^2) dx. \end{aligned} \quad (2.6)$$

By the Hölder and Sobolev embedding inequalities, we have

$$\begin{aligned} & \int (|u|^2 + |\nabla d|^2)(|\nabla u|^2 + |\nabla^2 d|^2) dx \\ & \leq C(\|u\|_6^2 + \|\nabla d\|_6^2)(\|\nabla u\|_2 + \|\nabla^2 d\|_2)(\|\nabla u\|_6 + \|\nabla^2 d\|_6) \\ & \leq C(\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2)(\|\nabla u\|_2 + \|\nabla^2 d\|_2)(\|\nabla^2 u\|_2 + \|\nabla^3 d\|_2). \end{aligned} \quad (2.7)$$

It follows from integrating by parts that

$$\begin{aligned} & \int (|\nabla u|^2 + |\nabla^2 d|^2) dx = \int (\partial_i u \cdot \partial_i u + \partial_{ij}^2 d \cdot \partial_{ij}^2 d) dx \\ & = - \int (u \cdot \Delta u + \partial_i d \cdot \partial_i \Delta d) dx \leq (\|u\|_2 + \|\nabla d\|_2)(\|\nabla^2 u\|_2 + \|\nabla^3 d\|_2), \end{aligned} \quad (2.8)$$

and thus, recalling (2.7), we have

$$\begin{aligned} & \int (|u|^2 + |\nabla d|^2)(|\nabla u|^2 + |\nabla^2 d|^2) dx \\ & \leq C(\|u\|_2 + \|\nabla d\|_2)(\|\nabla u\|_2 + \|\nabla^2 d\|_2)(\|\nabla^2 u\|_2^2 + \|\nabla^3 d\|_2^2) \\ & \leq Cm(t)(\|\nabla^2 u\|_2^2 + \|\nabla^3 d\|_2^2) \leq C\varepsilon_1(\|\nabla^2 u\|_2^2 + \|\nabla^3 d\|_2^2) \\ & \leq \frac{1}{2}(\|\nabla^2 u\|_2^2 + 2a\|\nabla^3 d\|_2^2), \end{aligned} \quad (2.9)$$

provided $m(t) \leq \varepsilon_1$, and ε_1 is small. Substituting this inequality into (2.6), one gets the second conclusion. \square

By the aid of the above proposition, we can establish the uniform estimates in time on the strong solutions of (1.1)-(1.2) as in the following proposition.

Proposition 2.2. *Let (u, d) be a strong solution to system (1.1)-(1.2) on $\mathbb{R}^3 \times (0, T)$. Then there is a small positive constant ε_0 , such that if*

$$(\|u_0\|_2 + \|\nabla d_0\|_2)(\|\nabla u_0\|_2 + \|\nabla^2 d_0\|_2) \leq \varepsilon_0,$$

then

$$\sup_{0 \leq s \leq t} (\|u\|_{H^1}^2 + \|\nabla d\|_{H^1}^2) + \int_0^t (\|\nabla u\|_{H^1}^2 + \|\nabla^2 d\|_{H^1}^2) ds \leq C(\|u_0\|_{H^1}^2 + \|\nabla d_0\|_{H^1}^2),$$

for any $t \in (0, T)$, where C is a positive constant independent of T .

Proof. Let $m(t)$ be the function as defined in Proposition 2.1. Recalling the regularity properties of strong solutions and the definition of $m(t)$, one can easily see that $m(t)$ is continuously increasing on $[0, T)$. By assumption, it is obvious that $m(0) \leq \varepsilon_0$. Define $T_0 = \sup\{t \in (0, T) \mid m(t) \leq \varepsilon_1\}$. By Proposition 2.1, we have

$$\begin{aligned} \frac{d}{dt} \int (|u|^2 + |\nabla d|^2 + 2W(d, \nabla d)) dx + \int (|\nabla u|^2 + a|\nabla^2 d|^2) dx &\leq 0, \\ \frac{d}{dt} \int (|\nabla u|^2 + |\nabla^2 d|^2) dx + \frac{1}{2} \int (|\nabla^2 u|^2 + a|\nabla^3 d|^2) dx &\leq 0, \end{aligned}$$

for any $t \in (0, T_0)$.

Integrating the above two inequalities in t , we have

$$\sup_{0 \leq s \leq t} (\|u\|_2^2 + \|\nabla d\|_2^2) + \int_0^t (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) ds \leq C(\|u_0\|_2^2 + \|\nabla d_0\|_2^2), \quad (2.10)$$

and

$$\sup_{0 \leq s \leq t} (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) + \int_0^t (\|\nabla^2 u\|_2^2 + \|\nabla^3 d\|_2^2) ds \leq C(\|\nabla u_0\|_2^2 + \|\nabla^2 d_0\|_2^2), \quad (2.11)$$

for any $t \in (0, T_0)$.

Then it follows from the above two estimates that, for any $t \in (0, T_0)$,

$$\begin{aligned} m(t) &\leq \sup_{0 \leq s \leq t} (\|u(s)\|_2 + \|\nabla d(s)\|_2) \sup_{0 \leq s \leq t} (\|\nabla u(s)\|_2 + \|\nabla^2 d(s)\|_2) \\ &\leq C(\|u_0\|_2 + \|\nabla d_0\|_2)(\|\nabla u_0\|_2 + \|\nabla^2 d_0\|_2) \leq C\varepsilon_0 \leq \frac{\varepsilon_1}{2}, \end{aligned}$$

by choosing $\varepsilon_0 \leq \frac{\varepsilon_1}{2C}$, where ε_1 is the constant in Proposition 2.1.

If $T_0 < T$, then the above estimates implies that there is a $T_1 > T_0$, such that

$$\sup_{0 \leq s \leq T_1} m(t) \leq \varepsilon_1,$$

from which, recalling the definition of T_0 , it has $T_1 \leq T_0$, contradicting to $T_1 > T_0$. Thus it must have $T_0 = T$, and consequently (2.10) and (2.11) hold for any $t \in (0, T)$. Therefore,

$$\sup_{0 \leq s \leq t} (\|u\|_{H^1}^2 + \|\nabla d\|_{H^1}^2) + \int_0^t (\|\nabla u\|_{H^1}^2 + \|\nabla^2 d\|_{H^1}^2) ds \leq C(\|u_0\|_{H^1}^2 + \|\nabla d_0\|_{H^1}^2),$$

proving the conclusion. \square

Proof of Theorem 1.1. Let ε_0 be the constant in Proposition 2.2, and suppose the conditions in Theorem 1.1 hold. By the local existence result, Lemma 2.1, there is a local strong solution (u, d) to system (1.1)–(1.2). Extend such local solution to the maximal existence time T^* . We are going to prove that $T^* = \infty$. Suppose, by contradiction, that $T^* < \infty$. By Proposition 2.2, it has

$$\sup_{0 \leq s \leq t} (\|u\|_{H^1}^2 + \|\nabla d\|_{H^1}^2) + \int_0^t (\|\nabla u\|_{H^1}^2 + \|\nabla^2 d\|_{H^1}^2) ds \leq C(\|u_0\|_{H^1}^2 + \|\nabla d_0\|_{H^1}^2),$$

for any $t \in (0, T^*)$. On account of this estimates uniform in time, one can apply the local existence result, Lemma 2.1, to extend (u, d) to a strong solution beyond T^* , contradicting to the definition of T^* . This contradiction implies that $T^* = \infty$, and thus (u, d) is a global strong solution. The uniqueness of global strong solutions is a direct consequence of that for local ones. This completes the proof. \square

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